Lecture 13

Plan: §7.2 Def^n of Laplace Transform

(Typo in the video, should be "§7.2", not "§7.4")

Q: Why "Laplace Transform"?

A: It will be a powerful tool to solve D.E. Before we see how to use it to solve D.E, we will spend 2-3 lectures as preparation.

Q: What is "Laplace transform":

A:

Def^n: Let $f(t)$ be a function on $[0,+\infty)$.

The Laplace tranform of f is the function F defined by

Note: "∞" means "$+\infty$"

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

Notation: $F(s) = \mathcal{L}\{f\}(s)$.

Curvy L

Remark: $\mathcal{L}\{f\}(s)$ might be defined for only some $s \in \mathbb{R}$, and not defined for other $s \in \mathbb{R}$

E.g Find the Laplace transform of

(1) $f(t) = e^{at}$, $t \geq 0$, a is a fixed real number.

(2) $g(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 5 \\ 0 & \text{if } t > 5 \end{cases}$

A: (1) $\mathcal{L}(f)(s) = \int_0^\infty e^{-st} e^{at}\, dt$

$$= \int_0^\infty e^{(a-s)t}\, dt$$

Recall from Calculus:

$$\int_0^\infty h(t)\,dt = \lim_{N\to\infty} \int_0^N h(t)\,dt$$

We have three cases:

(I) If $s=a$, then

$$\mathcal{L}(f)(s) = \int_0^\infty e^0\,dt$$

$$= \lim_{N\to\infty} \int_0^N 1\,dt$$

$$= \lim_{N\to\infty} N = \infty$$

Limit does not exist (DNE)

$\boxed{\text{Limit "exists" means the limit exists as a finite number.}}$

(II) If $s<a$, then

$a-s>0$

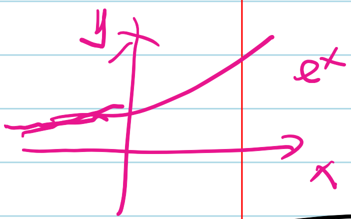
$$\mathcal{L}(f)(s) = \lim_{N\to\infty} \int_0^N e^{(a-s)t}\,dt$$

$N\to\infty \Rightarrow (a-s)N \to \infty$

$\Rightarrow e^{(a-s)N} \to \infty$

$$= \lim_{N\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^N$$

$$= \lim_{N\to\infty} \frac{1}{a-s}\left(e^{(a-s)N} - 1\right)$$



Limit DNE

(III) If $s>a$, then

$a-s<0$

$$\mathcal{L}(f)(s) = \lim_{N\to\infty} \int_0^N e^{(a-s)t}\,dt$$

when $N\to\infty \Rightarrow (a-s)N \to -\infty$

$\Rightarrow e^{(a-s)N} \to 0$

$$= \lim_{N\to\infty} \left(\frac{1}{a-s} e^{(a-s)t}\right)\Big|_0^N$$

$$= \lim_{N\to\infty} \frac{1}{a-s}\left(e^{(a-s)N} - 1\right)$$

Since $a-s<0$, $\Rightarrow \lim_{N\to\infty} e^{(a-s)N} = 0$

$$\Rightarrow \mathcal{L}(f)(s) = \frac{-1}{a-s} = \frac{1}{s-a}.$$

Hence $\mathcal{L}\{f\}(s) = \frac{1}{s-a}$, if $s>a$

$\mathcal{L}\{f\}(s)$ is not defined if $s \le a$.

(2) $\mathcal{L}(g)(s) = \int_0^\infty e^{-st} g(t)\, dt$

$$= \int_0^5 e^{-st} g(t) dt + \int_5^\infty e^{-st} g(t) dt$$

$$= \int_0^5 e^{-st} \cdot 1\, dt + \int_5^\infty 0\, dt$$

$$= \int_0^5 e^{-st} dt + 0$$

$$= \begin{cases} \int_0^5 1\cdot dt = 5; & \text{if } s=0 \\ \frac{1}{-s} e^{-st}\Big|_0^5 = \frac{1}{s} - \frac{e^{-5s}}{s}; & \text{if } s\neq 0 \end{cases}$$

$\int_5^\infty 0\, dt$

$= \lim_{N\to\infty} \int_5^N 0\, dt$

$= \lim_{N\to\infty}$

$\int e^{-st} dt = \frac{1}{-s} e^{-st} + C$ if $s \neq 0$

$\int e^{0} dt = \int 1 dt = t + C$ if $s=0$

Hence

$$\mathcal{L}(g)(s) = \begin{cases} 5 & \text{if } s=0 \\ \frac{1}{s} - \frac{e^{-5s}}{s} & \text{if } s\neq 0 \end{cases}$$

Thm: (Linearity of the Laplace transform)

Let f_1, f_2 be two functions, and assume their Laplace transforms $\mathcal{L}(f_1), \mathcal{L}(f_2)$ exist at $s_0 \in \mathbb{R}$. Let c_1, c_2 be constants. Then

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s_0) = c_1 \mathcal{L}\{f_1\}(s_0) + c_2 \mathcal{L}\{f_2\}(s_0)$$

In particular, $(c_1 = 1, \ c_2 = 1)$

$$\star \quad \mathcal{L}\{f_1 + f_2\}(s_0) = \mathcal{L}\{f_1\}(s_0) + \mathcal{L}\{f_2\}(s_0)$$

Q: When $\mathcal{L}(f)(s)$ exists?

A: We have some sufficient conditions to guarantee the existence of $\mathcal{L}(f)(s)$.
(might not be "necessary" conditions)

First, need two definitions.

Def^n 1: • A function $f(t)$ is said to be piecewise continuous on $[a,b]$ if f is continuous at every pt in $[a,b]$, except possibly for a finite number of points at which f has a jump discontinuity.

"f is continuous at t_0" $\Longleftrightarrow$

① f is defined at t_0

② $\lim_{t\to t_0^-} f(t) = \lim_{t\to t_0^+} f(t)$

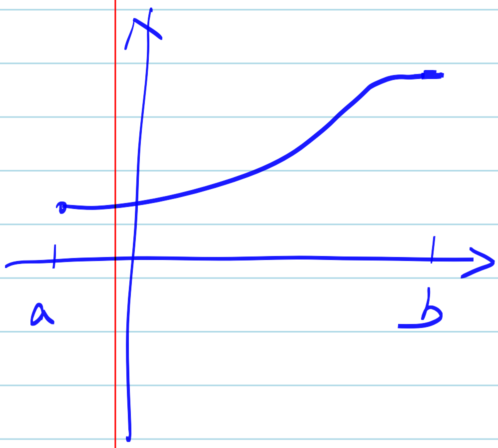
③ The above limits $= f(t_0)$

A jump discontinuity at t_0 $\Longleftrightarrow$ ② holds, but not all of ①, ②, ③ hold.

• f is called piecewise continuous on $[0,\infty)$ if f is piecewise continuous on $[0,N]$ for all $N>0$

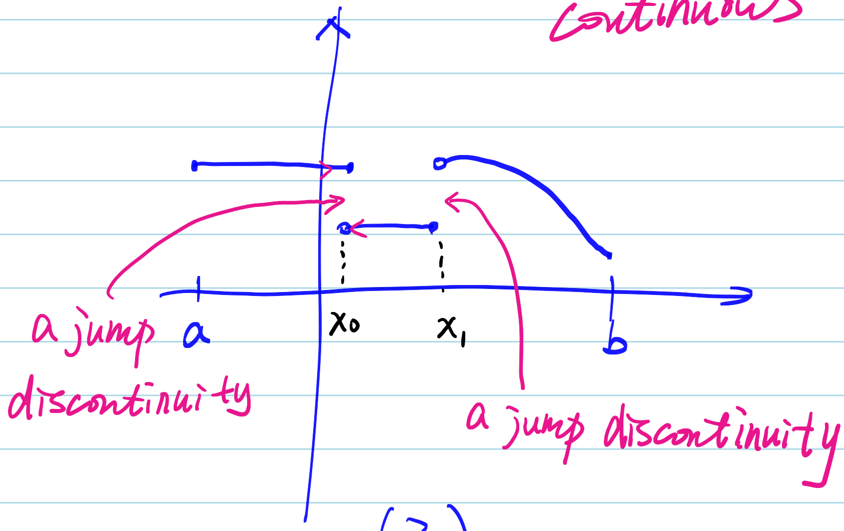
Remark: "continuous on $[a,b]$" $\Longrightarrow$ "piecewise continuous on $[a,b]$"

Continuous $\Rightarrow$ piecewise continuous



(1)

piecewise continuous

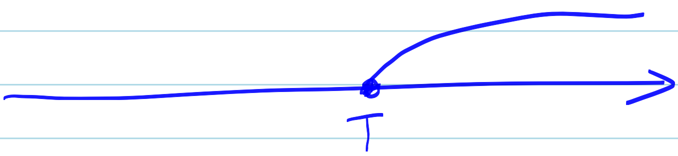


(2)

Def^n 2: A function $f(t)$ is said to be of exponential order α $(\alpha \in \mathbb{R})$ if there exist T, M such that

$$|f(t)| \leq M e^{\alpha t} \quad \text{for all } t \geq T$$

Intuitively, this means, from some point T, $|f(t)|$ grows no faster then $Me^{\alpha t}$ when $t \to \infty$.

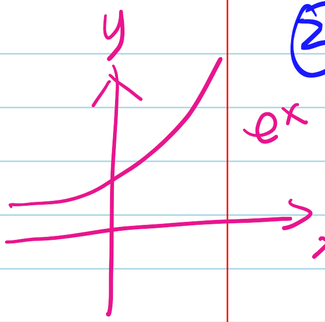


E.g ① $f(t) = e^{5t} \sin t$

f is of exponential order 5, as

$$|f| = |e^{5t} \sin t| = |e^{5t}|\,|\sin t| \leq |e^{5t}| = \underbrace{1}_{M} \cdot e^{5t}$$

for $\underbrace{t \geq 0}_{T}$.



② $f(t) = e^{t^2}$

f is NOT a function of any exponential order, as e^{t^2} grows faster than any $e^{\alpha t}$.

why? $t^2 > \alpha t$ for large enough t

Thm. If $f(t)$ is "piecewise continuous" on $[0, +\infty)$ and "of exponential order α", then

$$\mathcal{L}\{f\}(s) \text{ exists for } s > \alpha$$

E.g.:

$f(t) = e^{5t}\sin t$ is of exponential order $\underset{\alpha}{5}$

$\mathcal{L}\{f\}(s)$ exists for $s > 5$

Pf of Thm: We need to show

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$$

exists for $s > \alpha$.

Since f is of exponential order α, there exists T, M such that,

$$|f(t)| \le M e^{\alpha t} \quad \text{for } t \ge T.$$

Hence write

$$\int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^T e^{-st} f(t)\,dt}_{(\mathrm{I})} + \underbrace{\int_T^\infty e^{-st} f(t)\,dt}_{(\mathrm{II})}$$

For (I), note $e^{-st} f(t)$ is bounded on $[0,T]$ and $[0,T]$ is a finite interval.

$\Rightarrow$ (I) is finite

For (II), since on $[T, \infty)$,

$$|f(t)| \le M e^{\alpha t}$$

$$\Rightarrow \left| \int_T^\infty e^{-st} f(t)\,dt \right| \le \int_T^\infty e^{-st} |f(t)|\,dt \le \int_T^\infty e^{-st} M e^{\alpha t}\,dt$$

Note $e^{-st} M e^{\alpha t}$ is integrable on $[T, \infty)$ if $s > \alpha$

(means, $\int_T^\infty e^{-st} M e^{\alpha t}\,dt < \infty$)

why? Note if $s > \alpha$,

$$\int_T^\infty e^{-st} M e^{\alpha t}\,dt = M \int_T^\infty e^{(\alpha - s)t}\,dt$$

$$= M \lim_{N \to \infty} \int_T^N e^{(\alpha - s)t}\,dt$$

$$= M \lim_{N \to \infty} \frac{1}{\alpha - s} e^{(\alpha - s)t} \Big|_T^N$$

$$= M \frac{-e^{(\alpha - s)}}{\alpha - s} < \infty$$

$\Rightarrow$ (II) is finite.

☆ Important: Table of Laplace transform

$f(t)$	$\mathcal{L}\{f\}(s)$	Region of existence
c (a constant)	$\frac{c}{s}$	$s>0$
t^n, $n>0$ integer	$\frac{n!}{s^{n+1}}$	$s>0$
e^{kt}, $k \in \mathbb{R}$	$\frac{1}{s-k}$	$s>k$
$\sin bt$, $b \in \mathbb{R}$	$\frac{b}{s^2+b^2}$	$s>0$
$\cos bt$, $b \in \mathbb{R}$	$\frac{s}{s^2+b^2}$	$s>0$
$e^{at}\sin(bt)$, $a,b \in \mathbb{R}$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos(bt)$, $a,b \in \mathbb{R}$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$

E.g: Let $f(t) = \underset{c_1 f_1}{4t^2} + \underset{c_2 f_2}{5e^{3t}}$

Compute $\mathcal{L}\{f\}(s)$ for $s>3$.

Recall

if $\mathcal{L}\{f_1\}(s_0)$, $\mathcal{L}\{f_2\}(s_0)$ exist, $\Rightarrow$

$\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s_0)$

$= c_1 \mathcal{L}\{f_1\}(s_0) + c_2 \mathcal{L}\{f_2\}(s_0)$

Note $\mathcal{L}\{t^2\}(s)$ exists if $s>0$

$\mathcal{L}\{e^{3t}\}(s)$ exists if $s>3$.

$\Rightarrow$ If $s>3$, then

$\mathcal{L}\{t^2\}(s)$ and $\mathcal{L}\{e^{3t}\}(s)$ exist.

$\Rightarrow$ for $s>3$,

$$\mathcal{L}\{4t^2+5e^{3t}\}(s)$$

$$= 4\mathcal{L}\{t^2\}(s) + 5\mathcal{L}\{e^{3t}\}(s)$$

$$= 4\frac{2!}{s^3} + 5\frac{1}{s-3}$$

$$= \frac{8}{s^3} + \frac{5}{s-3}$$